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Kinetics of Random Sequential Parking on a Line

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We study the kinetics of irreversible random sequential parking of intervals of different sizes on an infinite line. For the simplest fixed-length parking distribution the model reduces to the known car-parking problem and we present an alternate solution to this problem. We also consider the general homogeneous case when the parking distribution varies as $x^{\alpha-1}$ at $x \ll 1$ with the length x of the filling interval. We develop a scaling theory describing such mixture-deposition processes and show that the scaled hole-size distribution $\Phi(\xi)$, with $\xi = xt^z$ a scaling variable, decays with the scaled mass ξ as $\xi^{-\theta} \exp(-\text{const} \cdot \xi^{1+\alpha})$ as $\xi \to \infty$. We determine scaling exponents z and θ, and find that at large times the coverage $\theta(t)$ has a power-law form $1 - \theta(t) \simeq t^{-\nu}$ with nonuniversal exponent $\nu = (2-\theta)/(1+\alpha)$ depending on the homogeneity index α.

KEY WORDS: Random sequential parking; hole-size distribution; scaling behavior.

1. INTRODUCTION AND MAIN RESULTS

Generation of random configurations of hard objects by successive additions is an interesting problem of statistical physics which appears naturally in an equilibrium version and in an irreversible version.[1] If the objects can move and if the time between the successive additions is large enough, the system of hard-core particles can equilibrate at constant density and the corresponding configurations will then be describable by equilibrium statistical mechanics. On the other hand, if the objects once inserted are clamped in their positions, nonequilibrium configurations are generated. The irreversible process corresponding to the latter situation is called random sequential adsorption (RSA). RSA processes have been used to

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model experimental situation in various fields ranging from ecology to chemistry and physics.[2] A number of related models have also been studied recently.[3] Understanding the kinetics of RSA processes is a challenging problem of considerable practical and theoretical interest.

RSA processes usually begin from an empty volume and continue until the jamming limit, that is, until it is impossible to place further objects. Like many statistical mechanical problems, exact solutions for the jamming limit and kinetics of the RSA processes exist only in one dimension.[4] In higher dimensions most of the information come from numerical work.[5-7] Many studies have been performed on objects of the same size with spherical symmetry (hard disks, hard spheres, etc.). For these cases it was first suggested by Feder[2] and later proven by Pomeau[8] and Swendsen[9] that in any dimension d, the coverage $\theta_d(t)$ close to the jamming limit $\theta_d(\infty)$ varies with time as $\theta_d(\infty)-\theta_d(t) \simeq t^{-\nu}$ with the exponent $\nu = 1/d$. Swendsen conjectured that this asymptotic behavior is universal for all isotropic RSA systems. However, there are numerical and analytical indications[7] that the Swendsen conjecture is not correct and the precise form of the convergence law depends on the shape and orientational freedom of depositing objects, interactions between adsorbed and adsorbing particles, etc.

Motivated by the above work, we look for the possible violation of the Swendsen conjecture in one-dimensional RSA of intervals of different sizes. The aim of the work presented here is to provide a general theoretical description of the evolution of the distribution of the hole sizes that results from random sequential parking of intervals on an infinite line. Such an extension of the RSA theory to a mixture of particles with a continuous distribution of sizes is not only of academic interest, since in many applications the adsorbing particles are indeed polydisperse (e.g., the adsorption of latex spheres on silica). However, rather little progress has been achieved in the theoretical treatment of such mixture-deposition processes. Most results were obtained numerically or approximately.[10-12] In the one-dimensional case, analytical results can be found for mixture-deposition models.[13-15] Most previous studies of the 1D deposition of mixtures were focused on the determination of jamming coverages.[13,14] In a very recent paper,[15] the kinetics of deposition of a two-component mixture of fixed-length and pointlike particles was investigated. From that paper[15] and also from other work[10] it follows that the Swendsen conjecture can be violated for mixtures and the long-time behavior is generally governed by the behavior of the parking distribution near a small-size cutoff.

We now state our main results. As in ref. 15, we consider a parking distribution with zero-size cutoff, but we study the general homogeneous case when the parking distribution varies as $x^{\alpha-1}$ at $x \ll 1$ with the length

x of the filling interval. One can see that the coverage is complete, $\theta_1(\infty) = 1$, and we will show that at large times the coverage follows the power law $1 - \theta_1(t) \simeq t^{-\nu}$ with the exponent ν depending on the parameter α of the model and generally distinct from Swendsen's value $\nu = 1$. An important feature of random sequential parking with zero-size cutoff is that a typical hole size decreases to zero during the parking process. The vanishing of the typical hole size is analogous to the vanishing of the inverse correlation length for a system near a second-order phase transition. Thus, at long times, one might anticipate that scaling and universality can be invoked to describe the nature of the hole-size distribution.

Our treatment is based on applying scaling to obtain asymptotic information about the kinetics of random sequential parking for a general class of models. We will argue that only a few details of the parking process are relevant in determining basic features of the hole-size distribution. Our approach parallels analogous developments in describing the kinetics of aggregation, coalescence, and fragmentation processes (see, e.g., ref. (16) and references therein).

The rest of this paper is organized as follows. In Section 2 we define the random sequential parking model and write the corresponding kinetic equations. For the simplest fixed-length parking distribution, $p(x) = \delta(x-1)$, we solve these equations exactly. The solution reproduces all known results[4] in a very simple manner. In Section 3 we consider homogeneous systems in which the parking distribution scales with the interval length as $x^{\alpha-1}$. We derive scaling solutions to the kinetic equations. The asymptotic forms of the hole-size distribution at small and large sizes are determined by the homogeneity index α. In Section 4 we discuss the existence of scaling, describe exact results for two simple models with $\alpha = 0$ and $\alpha = 1$, and compare these results with scaling predictions. We show that for these models solutions are indeed dominated by scaling ones. We expect that this feature is generally true for all homogeneous models, although our method of constructing exact results is applied only for integer α values. Finally, in the Appendix we present details of the calculations of the hole-size distribution at large sizes.

2. KINETIC EQUATIONS FOR RANDOM SEQUENTIAL PARKING ON A LINE

Let intervals be adsorbed sequentially and irreversibly onto an infinite line. Let $C(x, t)$ be the concentration of holes of length x at time t. Since the random sequential parking process takes place independently and homogeneously as a result of external sources, the evolution of $C(x, t)$ is described by the linear integrodifferential equation

$$\frac{\partial}{\partial t} C(x, t) = -C(x, t) \left[\int_0^x (x-z)\, p(z)\, dz \right] + 2 \int_x^\infty dy\, C(y, t) \int_0^{y-x} dz\, p(z) \tag{1}$$

The first term on the right-hand side of Eq. (1) accounts for the loss of holes of length x (to be termed x-voids) due to their covering by intervals of length z (to be termed by z-mers) with $z < x$, while the second term accounts for the gain of x-voids due to covering of holes with length larger than x.

First, we study the simplest car-parking problem, namely the covering of an infinite empty line by cars of identical length, $p(x) = \delta(x-1)$. In this case, Eq. (1) becomes

$$\frac{\partial}{\partial t} C(x, t) = -(x-1)\, C(x, t) + 2 \int_{x+1}^\infty dy\, C(y, t) \qquad \text{at} \quad x > 1 \tag{2a}$$

$$\frac{\partial}{\partial t} C(x, t) = 2 \int_{x+1}^\infty dy\, C(y, t) \qquad \text{at} \quad x < 1 \tag{2b}$$

One can observe that the obvious initial conditions

$$C(x, 0) = 0 \tag{3a}$$

are not enough. An analysis of the model on a ring of some finite length L shows that $C(x, 0) = L^{-1}\delta(x-L)$. Hence, one sees that $\int dx\, xC(x, 0) = 1$. Therefore, on the infinite line we must use the additional initial condition

$$\lim_{t \to 0} \int_0^\infty dx\, xC(x, t) = 1 \tag{3b}$$

Proceeding with a solution to the kinetic equation (2a), it is natural to test the following ansatz:

$$C(x, t) = A(t) \exp[-(x-1)\, B(t)] \tag{4}$$

at $x > 1$. The exponential form of the solution comes from the Poisson nature of the parking events. Initial conditions (3a) and (3b) imply

$$A(0) = B(0) = 0, \qquad \lim_{t \to 0} \frac{A(t)}{B^2(t)} = 1 \tag{5}$$

Substituting (4) into (2a) yields the reduced system of the ordinary differential equations

$$\frac{dA}{dt} = \frac{2A}{B}\exp(-B), \qquad \frac{dB}{dt} = 1 \tag{6}$$

Solving (6) subject to the initial data (5) gives

$$A(t) = t^2 F(t), \qquad B(t) = t \tag{7}$$

where we have introduced the shorthand

$$F(t) = \exp\left[-2\int_0^t \frac{1-\exp(-\tau)}{\tau}\,d\tau\right] \tag{8}$$

Combining (4) and (7), one can compute the integral on the right-hand side of Eq. (2b) and then find the x-void distribution at $x < 1$:

$$C(x, t) = 2\int_0^t d\tau\, \tau F(\tau)\exp(-x\tau) \tag{9}$$

From the exact solution (4), (7), and (9) we may obtain any feature of the car-parking process. For example, the time-dependent coverage

$$\theta_1(t) = 1 - \int_0^\infty dx\, xC(x, t) \tag{10}$$

may be expressed as follows:

$$\theta_1(t) = \int_0^t d\tau\, F(\tau) \tag{11}$$

A simple analysis shows that $\theta_1(t)$ has the following asymptotic behavior at large times:

$$\theta_1(t) = \theta_1(\infty) - \frac{\exp(-2\gamma)}{t} + \cdots \tag{12}$$

where $\gamma = 0.577215...$ is Euler's constant and $\theta_1(\infty)$ is the coverage at the jamming limit, $\theta_1(\infty) = \int_0^\infty dt\, F(t) = 0.747597....$

Thus we have reproduced known results for a fixed-length deposition model in a very simple manner; compare with the treatment of the same model by Gonzales, *et al.*[4] An important observation is that our approach is not affected if the deposition of finite-component mixtures is considered.

Indeed, for such mixtures, i.e., for the deposition of particles of different sizes L_j and rate values Q_j (here the index j numbers different species), the parking distribution becomes $p(x)=\sum_{j\geqslant 1} Q_j\delta(x-L_j)$ and after substituting this parking distribution into (1) one can see that the kinetic equations for x-void distribution functions will still be of the form (2). The only change will be more terms on the right-hand sides, with coefficients involving various rates and with lower limits in the integrals involving various sizes. An analysis of these equations (which is not presented here) shows that in the ordinary situation when the smallest size of the mixture has the same order as other sizes the kinetics of deposition at large times is governed by the smallest-size species and this kinetics only quantitatively differs from the one for the simplest fixed-length deposition model. In particular, Swendsen's conjecture is correct for such mixtures. In the anomalous situation when the least species has a considerably smaller size than all others, the deposition process proceeds on two distinct time scales. In the former stage, the smallest-size species may be considered as pointlike and the coverage approaches to the jamming limit as $\theta_1(\infty)-\theta_1(t)\simeq ct^{-b}\exp(-at)$ and this convergence law is nonuniversal (i.e., the constants a, b, and c depend on parameters of the model). In particular, for the two-component mixture our method reproduces recent results.[15] In the latter stage, the dimension of the least species becomes important and the exponential behavior crosses over to a slower power-law approach $\simeq t^{-1}$.

3. SCALING SOLUTIONS TO THE KINETIC EQUATIONS

As discussed above and also in recent studies,[11,12,15] interesting effects in the mixture-deposition kinetics are expected in situations when particle sizes differ significantly. Tarjus and Talbot[11] pointed out that kinetic behaviors of mixtures with a large number of species, approximated by a continuous distribution, may be also unexpected. We now combine these two sources of interesting deposition kinetics and consider mixtures with parking distribution functions $p(x)$ being continuous and positive for all x in some interval $0<x<l$. It is difficult to make general statements about the solutions to the kinetic equation (1) with such parking distribution functions. Therefore we consider homogeneous distributions, $p(x)\simeq x^{\alpha-1}$, where α is the homogeneity index [the normalization condition $\int dx\, p(x)=1$ implies that $\alpha>0$]. A basic reason for restricting ourselves to homogeneous parking distributions is that this choice includes most natural situations. Furthermore, the scaling ansatz reduces a two-variable problem to a single-variable problem, thus simplifying the description of parking kinetics. Finally, a scaling solution is universal in that it is independent of initial conditions.

The scaling ansatz for the hole-size distribution can be written as follows:

$$C(x, t) \simeq S^{-\theta} \Phi\left(\frac{x}{S(t)}\right) \tag{13}$$

where

$$S(t) \simeq t^{-z} \qquad \text{at} \quad t \gg 1 \tag{14}$$

is a typical hole size. It is assumed that this simple scaling picture becomes correct in the scaling region

$$x \ll 1, \qquad t \gg 1, \qquad \xi = x/S(t) = \text{finite} \tag{15}$$

As the precise form of the parking distribution we choose

$$p(x) = \begin{cases} \alpha x^{\alpha - 1} & \text{at} \quad x < 1 \\ 0 & \text{at} \quad x > 1 \end{cases} \tag{16}$$

A large-size cutoff is physically natural and cannot influence the behavior in the scaling region.

The kinetic equation (1) for the homogeneous parking distribution (16) becomes

$$\frac{\partial}{\partial t} C(x, t) = -\frac{x^{\alpha+1}}{\alpha+1} C(x, t) + 2 \int_x^\infty dy\, C(y, t)(y - x)^\alpha \tag{17}$$

Here we assume $x < 1$; the opposite case will be outlined later. Further, an error in the integral on the right-hand side of (17) [where $(y-x)^\alpha$ must be replaced by 1 at $y - x > 1$] is negligibly small in the scaling region.

We now find the exponents θ and z. Multiplying both sides of Eq. (17) by x^β and integrating over x, we find the following equation:

$$\frac{d}{dt} M_\beta(t) = \left(2 \frac{\Gamma(\alpha+1)\, \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} - \frac{1}{\alpha+1}\right) M_{\alpha+\beta+1}(t) \tag{18}$$

for the moments of the hole-size distribution

$$M_\beta(t) = \int_0^\infty dx\, x^\beta C(x, t) \tag{19}$$

In Eq. (18), Γ denotes the Euler gamma function. The moments $M_\beta(t)$ are connected with the moments m_β of the scaling function by the obvious relation

$$M_\beta(t) = S^{1+\beta-\theta}(t)\, m_\beta, \qquad m_\beta = \int_0^\infty d\xi\, \xi^\beta \Phi(\xi) \tag{20}$$

Choose now the exponent β in such a way that the numerical factor on the right-hand side of Eq. (18) vanishes, i.e.,

$$2\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} = \frac{1}{\alpha+1} \tag{21}$$

For this $\beta = \beta(\alpha)$, Eq. (18) implies that $M_\beta(t)$ does not depend on t. Hence, we immediately find the exponent θ from the relation (20), i.e.,

$$\theta = 1 + \beta \tag{22}$$

with the exponent β defined from the functional equation (21). Observe that the exponent θ depends on the parameter α of the model and does have a superuniversal value 2 as in aggregation and fragmentation processes.[16] This follows from the fact that the fraction of uncovered length,

$$\int_0^\infty dx\, xC(x, t) \simeq S^{2-\theta} \simeq t^{-z(2-\theta)} \tag{23}$$

approaches zero. Equation (23) implies that $\theta < 2$ and does not give a definite value of the exponent θ, unlike aggregation and fragmentation processes, where the mass conservation implies $\theta = 2$. A simple analysis shows that the transcendental equation (21) has only one positive solution $\beta = \beta(\alpha)$ and $\beta < 1$ for all $\alpha > 0$, i.e., $\theta = 1 + \beta$ is actually smaller than 2. We also mention three points: (i) β decreases when α increases; (ii) $\beta = 1 - \alpha + (2\pi^2/3 + 4\gamma^2 - 2 - 6\gamma)\,\alpha^2 + \cdots$ at $\alpha \ll 1$; and (iii) $\beta = (-3 + \sqrt{17})/2 = 0.5615288...$ at $\alpha = 1$, i.e., for the flat parking distribution $p(x) = 1$ for $x < 1$.

In order to determine the second exponent, we insert the scaling ansatz (13) into the kinetic equation (17). This allows one to separate the dependence on x and t into two scaling equations

$$\omega\left(\theta\Phi + \xi\frac{d\Phi}{d\xi}\right) = -\frac{\xi^{1+\alpha}}{1+\alpha}\Phi + 2\int_0^\infty d\eta\, \eta^\alpha \Phi(\eta+\xi) \tag{24a}$$

$$S^{-\alpha-2}\frac{dS}{dt} = -\omega \tag{24b}$$

where $\omega > 0$ is the separation constant. From Eq. (24b) we see that a typical hole size has the time dependence

$$S(t) \simeq [(1+\alpha)\,\omega t]^{-(1+\alpha)^{-1}} \qquad \text{at} \quad t \gg 1 \tag{25}$$

Hence, we find the exponent z,

$$z = (1+\alpha)^{-1} \tag{26}$$

It is now possible to confirm the violation of Swendsen's conjecture for the present homogeneous parking problem. Actually, the coverage follows the power-law asymptotics

$$1 - \theta_1(t) = \int_0^\infty dx\, xC(x, t) = S^{1-\beta}(t)\, m_1 \simeq t^{-(2-\theta)/(1+\alpha)} \tag{27}$$

with the exponent depending on the model parameter α. Figure 1 plots the exponent $v = (2-\theta)/(1+\alpha)$ versus α. Observe that in the interval $0 < \alpha < \alpha_c$ ($\alpha_c \approx 0.85$), the exponent v increases from $v = 0$ to $v_{\max} = v(\alpha_c) \approx 0.2198...$ and then at $\alpha > \alpha_c$ the exponent v decreases to zero.

Next we consider the properties of scaling solutions for small and large ξ. In the small-ξ limit, if one assumes that the scaling moments m_α, $m_{\alpha-1}$, etc., exist, then one can expand the integral on the right-hand side of Eq. (24a). This gives

$$\omega\left(\theta\Phi + \xi\frac{d\Phi}{d\xi}\right) = -\frac{\xi^{1+\alpha}}{1+\alpha}\,\Phi + 2m_\alpha - 2\alpha\xi m_{\alpha-1} + \cdots \tag{28}$$

Solving Eq. (28) yields the small-ξ expansion

$$\Phi(\xi) = \frac{2m_\alpha}{\omega\theta} - \frac{2\alpha m_{\alpha-1}}{\omega(\theta+1)}\,\xi + \cdots \qquad \text{at} \quad \xi \ll 1 \tag{29}$$

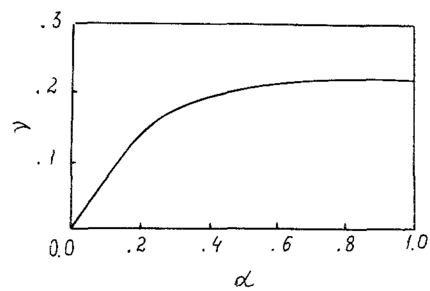

Fig. 1. The graph of the coverage exponent v versus the homogeneity parameter α.

Thus, up to a numerical factor the hole-size distribution decays as $C(x, t) \simeq t^{-(1+\beta)/(1+\alpha)}$ in the limit $x \ll t^{-1/(1+\alpha)}$.

In the large-ξ limit, an analogy with similar scaling solutions in a variety of fields (see, e.g., refs. 16 and 17) prompts us to test a skewed exponential

$$\Phi(\xi) \simeq \text{const} \cdot \xi^{-a} \exp(-b\xi^{c}) \qquad \text{at} \quad \xi \gg 1 \tag{30}$$

as a possible asymptotic. By inserting (30) into (24a) we obtain $a = \theta$, $b = \omega^{-1}(1+\alpha)^{-2}$, and $c = 1+\alpha$. So,

$$\Phi(\xi) \simeq \text{const} \cdot \xi^{-\theta} \exp[-\omega^{-1}(1+\alpha)^{-2}\,\xi^{1+\alpha}] \qquad \text{at} \quad \xi \gg 1 \tag{31}$$

A more rigorous derivation of the large-ξ asymptotic behavior (31) is given in the Appendix.

Finally, we describe the behavior beyond a large-size cutoff, i.e., at $x > 1$ in our units. This may be done for the general parking model with an arbitrary parking distribution $p(x)$. The kinetic equation (1) becomes

$$\begin{aligned}\frac{\partial}{\partial t} C(x, t) = &-(x-L)\, C(x, t) + 2 \int_{x+1}^{\infty} dy\, C(y, t) \\ &+ 2 \int_{x}^{x+1} dy\, C(y, t) \int_{0}^{y-x} dz\, p(z)\end{aligned} \tag{32}$$

where $L = \int_0^1 z p(z)\, dz$ is the average size of parked intervals.

To solve Eq. (32), we use the ansatz

$$C(x, t) = A(t) \exp[-(x-L)\, B(t)] \tag{33}$$

By inserting (32) into (33) and solving the resulting differential equations, we obtain

$$A(t) = t^2 F(t) \exp\left[2 \int_0^1 \frac{dy}{y} (1 - e^{-ty}) \int_0^y dz\, p(z)\right], \qquad B(t) = t \tag{34}$$

with $F(t)$ given by (8). In particular, for the homogeneous parking distribution (16) we obtain

$$C(x, t) = t^2 F(t) \exp\left[\frac{2}{\alpha} - \frac{2\Gamma(\alpha, t)}{t^{\alpha}}\right] \exp\left[-\left(x - \frac{\alpha}{\alpha+1}\right) t\right] \tag{35}$$

where $\Gamma(\alpha, t) = \int_0^t d\tau\, \tau^{\alpha-1} \exp(-\tau)$ is the incomplete gamma function.

Next we compute the fraction of holes with sizes lying beyond a cutoff

$$\rho(t) = \int_1^\infty dx\, xC(x, t) = (1+t)\, t^{-2} A(t) \exp[-(1-L)\, t] \tag{36}$$

In particular, at large times, for the homogeneous model (16) one has

$$\rho(t) \simeq (1+t)\, t^{-2} \left(1 - \frac{2\Gamma(\alpha)}{t^\alpha}\right) \exp\left(\frac{2}{\alpha} - 2\gamma - \frac{t}{1+\alpha}\right) \tag{37}$$

4. EXISTENCE OF SCALING

Here we discuss two crucial assumptions of the preceding analysis, the existence of scaling and the feature of a solution for arbitrary initial conditions to be dominated by the scaling solution at large times. We do not prove the existence of scaling rigorously, but show that the moments asymptotically approach a power law in accordance with the scaling behavior. First, remember that the moment $M_\beta(t)$, with β a solution of (21), approaches some constant C at large times. Then the moment relation (18) gives

$$M_{\beta-\alpha-1}(t) \simeq Ct \left[2 \frac{\Gamma(\alpha+1)\, \Gamma(\beta-\alpha)}{\Gamma(\beta+1)} - \frac{1}{\alpha+1}\right] \quad \text{at} \quad t \gg 1$$

Iterating this process, we arrive at the asymptotic solution

$$M_{\beta-k\alpha-k}(t) \simeq Ct^k (k!)^{-1} \prod_{j=1}^{k} \left(2 \frac{\Gamma(\alpha+1)\, \Gamma[\beta+1-j(\alpha+1)]}{\Gamma[\beta+1-(j-1)(\alpha+1)]} - \frac{1}{\alpha+1}\right) \quad \text{at} \quad t \gg 1 \tag{38}$$

for a discrete set of equidistant index values $\beta - k(\alpha+1)$. Assuming that the form of the moments for arbitrary index interpolates smoothly between moments defined on a discrete set, we find that Eq. (38) reproduces the asymptotic behavior of the moments for the system obeying scaling; cf. Eq. (38) with Eqs. (14) and (20).

To justify that an arbitrary solution may be dominated by the scaling solution, it is instructive to investigate explicitly solvable examples. The simplest such case is the limit $\alpha \to 0$ when the parking distribution (16) becomes $p(x) = \delta(x)$. Hence, an infinite line is subject to random cutting events. This model, called the random scission model,[18] is governed by the equation

$$\left(\frac{\partial}{\partial t} + x\right) C(x, t) = +2 \int_x^\infty dy\, C(y, t) \tag{39}$$

A general solution to this equation was found many years ago[19] and reads

$$C(x, t) = \exp(-xt) \left\{ C(x, 0) + \int_x^\infty dy\, C(y, 0)[2t + t^2(y-x)] \right\} \tag{40}$$

From (40) it can be seen that the scaling form exists if we keep $xt = \text{const}$ while taking the limit $x \to 0$ and $t \to \infty$. We find that the exact solution (40) approaches the scaling solution (apart from an overall constant factor), as expected:

$$C(x, t) = At^2 \exp(-xt), \qquad A = \int_0^\infty dy\, yC(y, 0) \tag{41}$$

The exponents $\theta = 2$ and $z = 1$ agree with our previous findings; see Eqs. (21), (22), and (26).

It would be useful to obtain exact solutions for homogeneous parking distributions with $\alpha > 0$ and to compare such solutions with the scaling ones. Some exact results can be found for integer α values, but we have not succeeded so far in solving Eq. (17) at arbitrary α, although the form of this equation is similar to some equations arising in the context of fragmentation.[18,20,21] Here we concentrate on the homogeneous parking distribution (16) with the homogeneity index $\alpha = 1$. We shall study the kinetics of parking subject to the monodisperse initial conditions $C(x, 0) = l^{-1}\delta(x - l)$, and for simplicity we shall assume that the initial size l is equal to the large-size cutoff of the parking distribution (16), $l = 1$. In this case the kinetics of covering is governed by the equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} x^2 \right) C(x, t) = 2 \int_x^1 dy\, C(y, t)(y - x) \tag{42}$$

We now find the moments $M_n(t) = \int_0^1 dx\, x^n C(x, t)$, with n a positive integer, following the method developed Charlesby[22] for the random scission model; see also Ziff and McGrady.[20] First, we rewrite the moment relation (18) as

$$\frac{d}{dt} M_n(t) = -\frac{(n-\beta)(n+\beta+3)}{2(n+1)(n+2)} M_{n+2}(t) \tag{43}$$

where $\beta = (-3 + \sqrt{17})/2 = 0.5615288\ldots$.

Using $M_n(0) = 1$ at all n (as follows from the monodisperse initial conditions) and iterating the moment relation (43), one can compute all derivatives of the moments at $t = 0$ and then obtain $M_n(t)$ from the Taylor series $M_n(t) = M_n(0) + tM'_n(0)/1! + t^2 M''_n(0)/2! + \dots$. The final exact expression for the moments is

$$M_n(t) = \left\{\frac{\Gamma[(n-\beta)/2]\,\Gamma[(n+\beta+3)/2]}{\Gamma[(n+1)/2]\,\Gamma[(n+2)/2]}\right\}^{-1}$$

$$\times \sum_{j=0}^{\infty} \frac{(-t/2)^j}{j!} \frac{\Gamma[j+(n-\beta)/2]\,\Gamma[j+(n+\beta+3)/2]}{\Gamma[j+(n+1)/2]\,\Gamma[j+(n+2)/2]}$$

$$= {}_2F_2\left[\frac{n-\beta}{2}, \frac{n+\beta+3}{2}; \frac{n+1}{2}, \frac{n+2}{2} \,\middle|\, -\frac{t}{2}\right] \tag{44}$$

where ${}_2F_2$ is the generalized hypergeometric function.

Turn now to the long-time limit. Considerable calculation yields the rather compact asymptotic result

$$M_n(t) \simeq \frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)} \frac{\Gamma(n+1)}{\Gamma((n+\beta+3)/2)} (2t)^{-(n-\beta)/2} \tag{45}$$

On the other hand, the scaling solution for this model has the form

$$C(x, t) \simeq (2t)^{(1+\beta)/2}\, \Phi[x(2t)^{1/2}] \tag{46}$$

Putting this result into the definition for the moments and comparing with the asymptotic formula (45), we find

$$\int_0^\infty d\xi\, \xi^n \Phi(\xi) = \frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)} \frac{\Gamma(n+1)}{\Gamma((n+\beta+3)/2)} \tag{47}$$

We see that in the long-time limit the solution relaxes to the scaling form (46). Moreover, we have computed the moments of the scaling function $\Phi(\xi)$. Although Eq. (47) has been derived for integer n values, we expect that this equation is generally true. Hence, the scaling function can now be obtained by computing the inverse Mellin transform. This yields

$$\Phi(\xi) = \frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)} \left(\frac{2}{\pi\xi}\right)^{1/2} \exp\left(\frac{-\xi^2}{8}\right) W_{-1/4-\beta/2,\,-1/4}\left(\frac{\xi^2}{4}\right) \tag{48}$$

where $W_{a,b}$ is the Whittaker function. From this exact scaling solution one can find the large-ξ behavior,

$$\Phi(\xi) \simeq \frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)} 2\pi^{-1/2} \xi^{-1-\beta} \exp\left(\frac{-\xi^2}{4}\right) \tag{49}$$

So, for the flat distribution, $\alpha=1$, we again confirm the large-ξ asymptotic behavior (31) and find numerical factors which remain undetermined in the scaling treatment.

In closing we note that it is possible to apply this method for constructing exact results in homogeneous models with other integer α values. However, the results become very cumbersome even for $\alpha=2$.

APPENDIX

Here we present a more rigorous derivation of the skewed exponential asymptotic behavior (31). The analysis which follows is based on the procedure described by Cheng and Redner.[16] In the first step we compute the "scaled" moments m_λ for a discrete set of equidistant λ values. Then we assume that the moments defined on a discrete set can be extended to all λ. Finally, we find the scaling function by computing the inverse Mellin transform.

We start with Eq. (24a) and convert it to a relation involving the moments of $\Phi(\xi)$ by multiplying both sides of (24a) by ξ^λ and integrating over all ξ. Thus we obtain the linear recursion relation for the moments of the scaling function:

$$m_{1+\alpha+\lambda}=\omega(1+\alpha)(\lambda-\beta)\left[1-2\frac{\Gamma(2+\alpha)\,\Gamma(1+\lambda)}{\Gamma(2+\alpha+\lambda)}\right]^{-1} \tag{A1}$$

From this equation one can compute the asymptotic form of m_λ for a discrete set of equidistant λ values. Because the large-ξ behavior of the scaling function $\Phi(\xi)$ corresponds to m_λ for large values of λ, we choose $\lambda=k(1+\alpha)$, with k a positive integer, iterate (A1), and find

$$\begin{aligned} m_{k+k\alpha}=m_{1+\alpha}[\omega(1+\alpha)^2]^{k-1}\prod_{n=1}^{k-1}\left(n-\frac{\beta}{1+\alpha}\right)\\ \times\left\{1-\frac{2\Gamma(2+\alpha)\,\Gamma[1+n(1+\alpha)]}{\Gamma[1+(n+1)(1+\alpha)]}\right\}^{-1} \end{aligned} \tag{A2}$$

Making use the asymptotics

$$\frac{\Gamma[1+n(1+\alpha)]}{\Gamma[1+(n+1)(\alpha+1)]}\propto n^{-1-\alpha} \qquad \text{at} \quad n\gg 1$$

$$\prod\left(1-\frac{\text{const}}{n^{1+\alpha}}\right)\propto 1$$

$$\prod_{n=1}^{N}\left(1-\frac{\alpha}{n}\right)\propto N^{-\alpha} \qquad \text{at} \quad N\gg 1$$

we transform (A2) into

$$m_{k+k\alpha} \propto [\omega(1+\alpha)^2]^k \, k! \, k^{-1-\beta/(1+\alpha)} \tag{A3}$$

Denoting $\mu = k(1+\alpha)$ and employing Stirling's approximation, we obtain

$$m_\mu \propto [\omega(1+\alpha)\, e^{-1}\mu]^{\mu/(1+\alpha)} \, \mu^{-\beta/(1+\alpha)-1/2} \qquad \text{at} \quad \mu \gg 1 \tag{A4}$$

Noting that the scaling moments are just the Mellin transform of the scaling function [see the definition (20)] and computing the inverse Mellin transform of m_μ given by Eq. (A4), we again arrive at the asymptotic behavior (31). Thus we conclude that the scaling function $\Phi(\xi)$ has the universal asymptotic form (31) at large ξ, for arbitrary homogeneous parking distributions.

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